M.S. THESIS

Infrared divergence for Soft Graviton

중력자 상호작용에서 발생하는 적외선 발산

BY

Kim, SoDam

2014 년 2 월

DEPARTMENT OF PHYSICS AND
ASTRONOMY
COLLEGE OF NATURAL SCIENCE
SEOUL NATIONAL UNIVERSITY

M.S. THESIS

Infrared divergence for Soft Graviton

중력자 상호작용에서 발생하는 적외선 발산

BY

Kim, SoDam

2014 년 2 월

DEPARTMENT OF PHYSICS AND
ASTRONOMY
COLLEGE OF NATURAL SCIENCE
SEOUL NATIONAL UNIVERSITY

Infrared divergence for Soft Graviton

중력자 상호작용에서 발생하는 적외선 발산

지도교수 이 수 종

이 논문을 이학석사 학위논문으로 제출함

2013 년 12 월

서울대학교 대학원

물리천문학부

김 소 담

김 소 담의 이학석사 학위논문을 인준함

2013 년 12 월

위 원 장	김 형 도	(인)
부위원장	이 수 종	(인)
위 원	이 원 종	(인)

Abstract

In the length scale more than Plank length($1.616 \times 10^{-33} cm$), we describe graviton as a spin 2 tensor boson, and we use Feynman Rules. In this article, it is shown that the infrared divergences arising in the quantum theory of graviton can be removed by using methods of quantum electrodynamics(QED). In the massless electrodynamics, such as Yang-Mills theory, the infrared divergences from the case, external particle is massless, can not be removed. However we verify the cancellation of divergences occurs dealing with graviton, that is caused by the particular characteristic of gravitational coupling. We estimate the gravitational radiation emitted during thermal collisions in the sun. This is review paper for [Infrared Photons and Gravitons, S. Weinberg, phys. Rev. 140, B516].

Keywords: Infrared divergence, Soft graviton, Gravitational radiation

Student Number: 2011-20390

Contents

List of Figures

Chapter 1

Introduction

We do not have the satisfactory quantum theory of gravitation. However, in this review paper, we describe graviton as a spin 2 tensor boson, and we use the method of quantum electrodynamics. As the infrared divergence from soft photon can be removed through the radiative corrections, we verify that the infrared divergence problem in the quantum theory of gravitation can be solved by using the same method in QED.

It is not surprising to solve the problem removing the infrared divergence, because we already know that the coupling between the soft graviton and other particles, external graviton line wave function, and internal graviton line propagator. Simply using these apparatuses, we can get the solution.

Although we use the same manners in quantum electrodynamics, there is a difference dealing with the problem when external particle is massless. In massless electrodynamics, such as Yang and Mills's, the cancellation of divergences does not occur. However, we can show the cancellation for infrared gravitons summing all such diagrams. This is caused by the characteristic of gravitational coupling.

When we treat the real soft-graviton emission, we get a formula for a emission rate and the spectrum of soft gravitons in arbitrary processes. Using this formula, we will estimate the gravitational radiation emitted from the thermal collisions in the sun.

Chapter 2

Real and Virtual Infrared Divergence

In this chapter, we treat infrared divergences for real soft graviton and virtual graviton. We only treat the case when all external particles are massive, and then in the next chapter we verify the massless external-particle case. As mentioned in introduction, we use the method of QED, that is, the method of treating infrared photon. In conclusion, we can see that the divergence occurs dealing with soft bremsstrahlung for real soft graviton and virtual graviton, each. However, when we consider those two cases, simultaneously, the infrared divergences are removed.

2.1 One Soft Graviton

In this section, we consider the case one soft-graviton is emitted from the scattering process. If we attach a soft-graviton line with momentum q to the outgoing external line, terms of the one extra vertex for the transition $p + q \to p$ and the one extra propagator with momentum $p+q$ should be supplied. When a soft-graviton line is attached to the incoming external particle, the extra vertex for the transition $p \to p-q$ and the extra propagator with momentum $p-q$ are multiplied. The Fig 2.1 shows those Feynman diagrams.

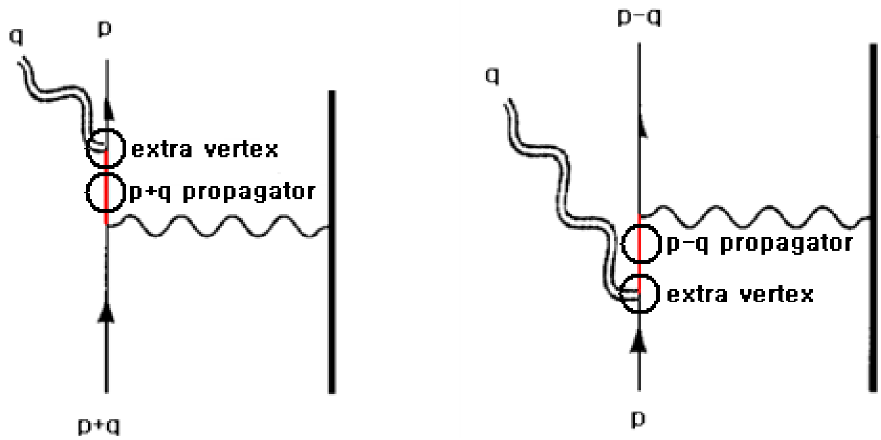

Figure 2.1: One soft graviton - One soft graviton is emitted from the external particle

When the external particle has zero spin, the extra factor is

$$\frac{1}{2}i(2\pi)^4(8\pi G)^{1/2}(2p^\mu + \eta q^\mu)(2p^\nu + \eta q^\nu) \times [-i(2\pi)^{-4}][(p+\eta q)^2 + m^2 - i\epsilon]^{-1} \quad (2.1)$$

where $\eta = +1$ or -1 for an outgoing or incoming particle. μ, ν are the polarization indices of graviton. In the limit $q \to 0$, Eq. 2.1 becomes

$$(8\pi G)^{1/2}\eta p^\mu p^\nu/[p \cdot q - i\eta\epsilon]. \quad (2.2)$$

The limiting form Eq. 2.2 is valid for any spin. For instance, the external particle has spin 1/2, then the extra factor is

$$-\frac{1}{4}(2\pi)^4(8\pi G)^{1/2}[(2p^\mu+q^\mu)\gamma^\nu+(2p^\nu+q^\nu)\gamma^\mu] \times[-i(2\pi)^{-4}][\frac{-i(p^\lambda+q^\lambda)\gamma_\lambda+m}{(p+q)^2+m^2-i\epsilon}]. \quad (2.3)$$

The Dirac spinor $\bar{u}$ satisfies Dirac Equation ($\bar{u}[ip^\lambda\gamma_\lambda+m]=0$). So when we move the propagator numerator to the left of the vertex function, the equation is equal to Eq. 2.2 in the limit $q\to 0$. We can derive the Eq. 2.2 for any spin considering Lorentz invariance.

In the limit $q\to 0$, $1/(p\cdot q)$ term is dominant, so the general form of the factor is

$$(8\pi G)^{1/2}\sum_n \eta_n p_n^\mu p_n^\nu/[p_n\cdot q-i\eta_n\epsilon] \quad (2.4)$$

where n is the label of external particles. These transitions are independent, so we just sum the factors.

2.2 Many Soft Gravitons

Let's consider the case that the soft gravitons are emitted from an outgoing external particle. As shown in the Fig. 2.2, the graviton with momentum q_r is attached to the last, and the graviton with momentum q_s is next to the graviton r(where $r,s=1,2,3,...$).

When we treat this situation, the dominator of Eq. 2.2 becomes $[p\cdot q_r-i\eta\epsilon]^{-1}[p\cdot(q_r+q_s)-i\eta\epsilon]^{-1}\cdots$

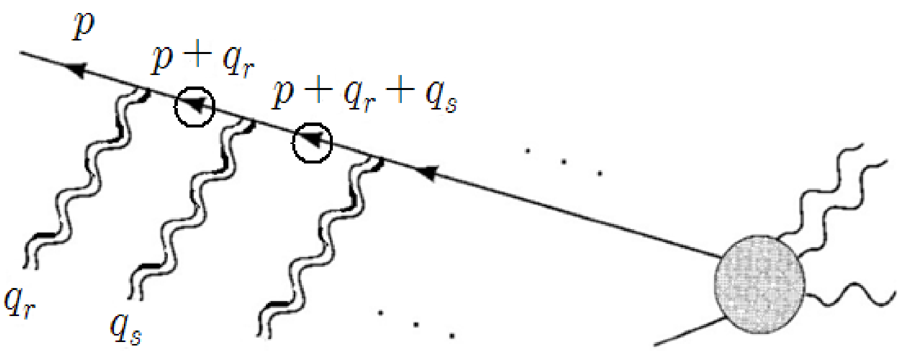

Figure 2.2: Many soft gravitons - Many soft gravitons are emitted from the outgoing external particle

For general discussion, let us consider the N soft gravitons emitted from an outgoing external particle. Then we must sum all $N!$ permutations. Then the dominator becomes

$$[p \cdot q_1 - i\eta\epsilon]^{-1}[p \cdot q_2 - i\eta\epsilon]^{-1} \cdots [p \cdot q_N - \eta\epsilon]^{-1}. \tag{2.5}$$

For example, $N = 2$:

$[p \cdot q_1 - i\eta\epsilon]^{-1}[p \cdot (q_1 + q_2) - i\eta\epsilon]^{-1} + [p \cdot q_2 - i\eta\epsilon]^{-1}[p \cdot (q_2 + q_1) - i\eta\epsilon]^{-1}$
$= [p \cdot q_1 - i\eta\epsilon]^{-1}[p \cdot q_2 - i\eta\epsilon]^{-1}$

For general N, this can be proved by mathematical induction. Likewise, when the N soft gravitons are emitted from other external legs, that factor is same as Eq. 2.2. The complete form for the emitted N soft gravitons from an external particle is

$$(8\pi G)^{1/2}\eta p^{\mu}p^{\nu} \times [p\cdot q_1 - i\eta\epsilon]^{-1}[p\cdot q_2 - i\eta\epsilon]^{-1}\cdots[p\cdot q_N - \eta\epsilon]^{-1} \tag{2.6}$$

2.3 Virtual Infrared Divergences

For discussion of infrared or soft graviton, we should set up the upper limit of graviton's energy. 'Soft' means 'undetectable,' so if the bottom energy limit of determination is Λ_{IR}, in our discussion we treat the graviton whose energy is less than the energy, Λ_{IR}, that is, $|q| \le \Lambda_{IR}$. Also, we impose the bottom limit of graviton's energy, λ, in order to see the logarithmic divergence for $\lambda \to 0$. The divergence occurs through the power series of $ln\lambda$ for $\lambda \to 0$.

The virtual graviton can be described by connecting two real soft-graviton lines. When pairing the real-soft gravitons, we should supply the graviton propagator. If we consider two external legs that have N soft-graviton lines, each, we can make the N-pair virtual infrared-graviton line. For pairing the infrared gravitons, the propagator term should be multiplied to each pair. The effective graviton propagator joinng a $(\mu\nu)$ vertex with a $(\rho\sigma)$ vertex is known to be

$$\frac{-i}{2(2\pi)^4}\frac{(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma})}{q^2 - i\epsilon}. \tag{2.7}$$

We should sum over polarization indices and integrate over q's. Also we should divide by $2^N N!$, because virtual gravitons are not distinguishable, so dividing $N!$. The factor 2^N is the term about the directions of flowing momentum through virtual gravitons. In conclusion,

$$\frac{1}{N!}[\frac{1}{2}\int_{\lambda}^{\Lambda_{IR}} d^4q B(q)]^N \tag{2.8}$$

where

$$B(q) = \frac{-8\pi G i}{(2\pi)[q^2 - i\epsilon]} \times \sum_{n,m} \frac{\eta_n \eta_m (p_n \cdot p_m)^2 - \frac{1}{2}m_n^2 m_m^2}{[p_n \cdot q - i\eta_n \epsilon][-p_m \cdot q - i\eta_m \epsilon]}. \tag{2.9}$$

$B(q)$ is the result of joining a pair of factors (2.7) with a graviton propagator.

In Eq. 2.9, we should notice $-p_m \cdot q$ term, not $p_m \cdot q$ in the second denominator, because if soft graviton with momentum q is emitted from external line 'n', this should be absorbed to external line 'm.' Summing over N, S matrix for any process is expressed to following

$$S_{\beta\alpha} = S^0_{\beta\alpha} exp[\frac{1}{2} \int_\lambda^{\Lambda_{IR}} d^4 q B(q)] \tag{2.10}$$

where $S^0_{\beta\alpha}$ is S matrix without infrared virtual gravitons.

The rate for $\alpha \to \beta$ is the absolute square of Eq. 2.10

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha} exp[Re \int_\lambda^{\Lambda_{IR}} d^4 q B(q)]. \tag{2.11}$$

The real part of integral is comes only from the $i\pi\delta(q^2)$ term in the graviton propagator, so

$$Re \int_\lambda^{\Lambda_{IR}} d^4 q B(q) = -\frac{-8\pi G}{2(2\pi)^3} \int_\lambda^{\Lambda_{IR}} d^4 q \delta(q^2) \times \sum_{n,m} \frac{\eta_n \eta_m (p_n \cdot p_m)^2 - \frac{1}{2}m_n^2 m_m^2}{[p_n \cdot q][p_m \cdot q]}$$
$$= -B ln(\Lambda_{IR}/\lambda) \tag{2.12}$$

where B is the dimensionless constant.

$$B \equiv \int d^2\Omega B(\hat{q}) \tag{2.13}$$

where

$$B(\hat{q}) \equiv \frac{8\pi G}{2(2\pi)^3} \sum_{n,m} \frac{\eta_n \eta_m (p_n + p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{[E_n - \vec{P_n} \cdot \hat{q}][E_m - \vec{P_m} \cdot \hat{q}]} \tag{2.14}$$

Calculating Eq. 2.13,

$$B = \frac{G}{2\pi} \sum_{n,m} \eta_n \eta_m m_n m_m \frac{1 + \beta_{nm}^2}{\beta_{nm}(1 - \beta_{nm}^2)^{1/2}} ln(\frac{1 + \beta_{nm}}{1 - \beta_{nm}}) \tag{2.15}$$

where β_{nm} is the relative velocity of particles n and m in the rest frame of either.

$$\beta_{nm} \equiv [1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}]^{1/2} \tag{2.16}$$

In this calculation, we see that B is the positive dimensionless constant. Using Eq. 2.11, and Eq. 2.12, we can see the cutoff-dependence of rate.

$$\Gamma_{\beta\alpha} = \Gamma_{\beta\alpha}^0 (\lambda / \Lambda_{IR})^B \tag{2.17}$$

Because $B > 0$, $\Gamma_{\beta\alpha}$ becomes zero in the limit $\lambda \to 0$. This is paradoxial situation, and this paradox can be solved by considering the infrared divergence from the bremsstrahlung of soft real graviton.

2.4 Real Infrared Divergence

S matrix for emitting N real soft-graviton lines can be expressed by suppling N-factor form of Eq. 2.7 to the non-radiative S matrix for $\alpha \to \beta$ without real soft-graviton. And then considering the appropriate graviton wave function, those factors are contracted. The appropriate graviton wave function is following

$$(2\pi)^{-3/2}(2|\vec{q}|)^{-1/2}\epsilon^*_\mu(\vec{q},\pm 1)\epsilon^*_\nu(\vec{q},\pm 1) \tag{2.18}$$

where $\vec{q}$ is graviton momentum, $h - \pm 2$ is its helicity, and ϵ_μ is polarization vector. Then graviton emission-matrix element is following

$$S^{gr}_{\beta\alpha}(12\cdots N) = S_{\beta\alpha}\prod_{r=1}^{N}(2\pi)^{-3/2}(2|\vec{q}|)^{-1/2}(8\pi G)^{1/2} \times \sum_n \frac{\eta_n[p_n \cdot \epsilon^*(\vec{q_r}, \frac{1}{2}h_r)]^2}{[p_n \cdot q_r]}. \tag{2.19}$$

In case that the N emitted gravitons have the momenta near $\vec{q_1}\cdots\vec{q_N}$, the rate for $\alpha \to \beta$ is given by squaring Eq. 2.19, summing over helicities, and dividing by $N!$, because gravitons are bosons. Then

$$\Gamma^{gr}_{\beta\alpha}(\vec{q_1}\cdots\vec{q_N})d^3q_1\cdots d^3q_N = \frac{1}{N!}\Gamma_{\beta\alpha}\prod_{r=1}^{N}\beta(\vec{q_r})d^3q_r \tag{2.20}$$

where $\Gamma_{\beta\alpha} = |S_{\beta\alpha}|^2$.

And

$$\beta(\vec{q}) = (2\pi)^{-3}(2|\vec{q}|)^{-1}(8\pi G)\sum_{n,m}\frac{\eta_n\eta_m p_n^\mu p_n^\nu p_m^\rho p_m^\sigma \Pi_{\mu\nu\rho\sigma}(\vec{q})}{(p_n \cdot q)(p_m \cdot q)} \tag{2.21}$$

where $\Pi_{\mu\nu\rho\sigma}$ is polarization sum,

$$\Pi_{\mu\nu\rho\sigma}(\vec{q}) = \sum_\pm \epsilon_\mu(\vec{q},\pm)\epsilon_\nu(\vec{q},\pm)\epsilon^*_\rho(\vec{q},\pm)\epsilon^*_\sigma(\vec{q},\pm). \tag{2.22}$$

We know that

$$\Pi_{\mu\nu\rho\sigma}(\vec{q}) = \frac{1}{2}[\Pi_{\mu\rho}(\vec{q})\Pi_{\nu\sigma}(\vec{q}) + \Pi_{\mu\sigma}(\vec{q})\Pi_{\nu\rho}(\vec{q}) - \Pi_{\mu\nu}(\vec{q})\Pi_{\rho\sigma}(\vec{q})] \tag{2.23}$$

where

$$\Pi_{\mu\nu}(\vec{q}) = g_{\mu\nu} + q_\mu \lambda_\nu + q_\nu \lambda_\mu \tag{2.24}$$

$$\lambda^\mu \equiv (-\vec{q}, |\vec{q}|/2|\vec{q}|^2). \tag{2.25}$$

The $q\lambda$ terms in Π do not contribute, because energy and momentum are conserved

$$q_n \sum_n \eta_n p_n^\mu p_n^\nu / (p_n \cdot q) = \sum_n \eta_n p_n^\nu = 0 \tag{2.26}$$

So, the $g_{\mu\nu}$ term in Π is only effective, and Eq. 2.21 becomes

$$\beta(\vec{q}) = (2\pi)^{-3}(2|\vec{q}|)^{-1}(8\pi G) \sum_{n,m} \frac{\eta_n \eta_m (p_n \cdot p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(p_n \cdot q)(p_m \cdot q)} \tag{2.27}$$

or

$$\beta(\vec{q}) = B(\hat{q})/|\vec{q}|^3. \tag{2.28}$$

The rate for emitting gravitons with energies near $w_1 \cdots w_N$ is given by integrating Eq. 2.20 over solid angle. Using 2.28, and 2.13, we find

$$\Gamma^{gr}_{\beta\alpha}(w_1 \cdots w_N) dw_1 \cdots dw_N = \frac{B^N}{N!} \Gamma_{\beta\alpha} \frac{dw_1}{w_1} \cdots \frac{dw_N}{w_N}. \tag{2.29}$$

As showed in Eq. 2.29, the rate for emitting N soft-graviton has infrared divergence. For displying these divergences quantitatively, we calculate the rate for $\alpha \to \beta$ of any number of soft gravitons with total energy less than E, and with each energy w_r greater than the infrared cutoff λ. Using a step function representation, the rate can be expressed by following

$$\Gamma_{\beta\alpha}(\leq E) = \frac{1}{\pi}\sum_{N=0}^{\infty}\int_{\lambda}^{E} dw_1 \cdots \int_{\lambda}^{E} dw_N \int_{-\infty}^{\infty} d\sigma \frac{sinE\sigma}{\sigma} \times exp[i\sigma \sum_r w_r]\Gamma_{\beta\alpha}(w_1 \cdots w_N). \quad (2.30)$$

When applying Eq. 2.30 to Eq. 2.29, the graviton emission rate is

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{sinE\sigma}{\sigma} exp[B \int_{\lambda}^{E} \frac{dw}{w} e^{iw\sigma}] d\sigma \Gamma_{\beta\alpha} \quad (2.31)$$

For $\lambda \to 0$, the integrations over w become

$$\int_{\lambda}^{E} \frac{dw}{w} e^{iw\sigma} \to ln(E/\lambda) + \int_{0}^{E} \frac{dw}{w}(e^{iw\sigma} - 1) + \vartheta(\lambda) \quad (2.32)$$

So for $\lambda \to 0$, Eq. 2.31 becomes

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = (E/\lambda)^B b(B)\Gamma_{\beta\alpha} \quad (2.33)$$

where $b(x)$ is defined as

$$b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\sigma \frac{sin\sigma}{\sigma} exp[x \int_{0}^{1} \frac{dw}{w}(e^{iw\sigma} - 1)] \quad (2.34)$$

$$b(x) \cong 1 - \frac{1}{12}\pi^2 x^2 + \cdots \quad (2.35)$$

Since B is positive, the $(E/\lambda)^B$ term diverges for $\lambda \to 0$.

2.5 Cancellation of Divergences

In Eq. 2.17 the rate for $\alpha \to \beta$ considering vertex correction, that is, including infrared virtual graviton becomes zero, and this is paradoxial. But inserting Eq. 2.17 into Eq. 2.33 then all dependence on the infrared cutoff λ disappears. The conclusion is

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = (E/\Lambda)^B b(B) \Gamma^0_{\beta\alpha} \tag{2.36}$$

Λ was used to define what we mean by 'infrared' (recall, in Section 3, Λ was written as Λ_{IR}.) But if we change Λ to Λ' renormalizing $\Gamma^0_{\beta\alpha}$ by a factor $(\Lambda'/\Lambda)^B$, and then the conclusion does not change. This implies that we can get the same conclusion how we fix Λ except for estimating $\Gamma^0_{\beta\alpha}$ by ignoring all radiative corrections. So we could setup Λ as a ultraviolet cutoff, and fixing it to some mass typical of a particles in the transition $\alpha \to \beta$ is good strategy.

Through the term E^B in Eq. 2.36 we can get the energy spectrum for graviton ranging from zero up to smaller than the energy characterizing the reaction $\alpha \to \beta$, that is, $0 \leq E < \Lambda$.

Chapter 3

Graviton Emission From Massless-Particle Lines

In section 2, we treat only the emission of soft-graviton from massive external particle. In this section, we consider when the external particle has zero mass. The divergent term vanish in quantum theory of gravitation, but in quantum electrodynamics the cancellation does not occur. However, the massless and no-charged particle does not exist, so we do not have to think about that case considering photon emission from massless external-line. In gravitation theory, the massless particle have gravitational interaction with soft graviton, that is, the soft graviton can be emitted from the massless external-line. So we should consider that case. The divergence occurs because the denominator factors $(p \cdot q)$ in Eq. 2.14 will vanish for $\vec{q}$ parallel to $\vec{p}$ if p^2 is zero. However, we can see the cancellation of divergence.

Suppose that mass m_1 of particle labeled 1 vanishes, with fixed $\vec{p_1}$. Then by the binomial theorem the relative velocity Eq. 2.16 can be written as

$$\beta_{1n} = 1 - m_1^2 m_n^2 / 2(p_1 \cdot p_n)^2 + \vartheta(m_1^4) \tag{3.1}$$

where $p_1 = (\vec{p_1}, |\vec{p_1}|)$

Inserting Eq. 3.1 into Eq. 2.15, B can becomes up to $\vartheta(m_1^2)$

$$B = -\frac{\eta_1 G}{\pi} \sum_{n \neq 1} \eta_n (p_n \cdot p_1) ln[\frac{4(p_1 \cdot p_n)^2}{m_1^2 m_m^2}] \\ - \frac{G}{2\pi} \sum_{n,m \neq 1} \eta_n \eta_m (p_n \cdot p_m)(1 + \beta_{nm}^2) ln(\frac{1 + \beta_{nm}}{1 - \beta_{nm}}) \tag{3.2}$$

In the limit $m_1 \to 0$, the logarithmically divergent term is

$$+\frac{2\eta_1 G}{\pi} (ln m_1) \sum_{n \neq 1} \eta_n (p_n \cdot p_1) \tag{3.3}$$

Using energy and momentum conservation, the Eq. 3.3 becomes

$$+\frac{2\eta_1 G}{\pi} (ln m_1) \sum_{n \neq 1} \eta_n (p_n \cdot p_1) = -\frac{2G}{\pi} (ln m_1) p_1^2 = 0 \tag{3.4}$$

where $p_1^2 = 0$

Likewise, we can show the cancellation for several massless-particles. Through this section, we verify that there is a cancelation of these divergences for gravitons.

Chapter 4

Gravitational Radiation In Non-Relativistic Collisions

In section 2, the rate $\Gamma(\leq E)$ for a collision with radiated energy is

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = (E/\Lambda)^B b(B)\Gamma^0_{\beta\alpha} \tag{4.1}$$

where

$B = \frac{G}{2\pi}\sum_{n,m}\eta_n\eta_m m_n m_m \frac{1+\beta_{nm}^2}{\beta_{nm}(1-\beta_{nm}^2)^{1/2}} ln(\frac{1+\beta_{nm}}{1-\beta_{nm}})$

$b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\sigma \frac{sin\sigma}{\sigma} exp[x\int_0^1 \frac{dw}{w}(e^{iw\sigma}-1)]$

$b(x) \cong 1 - \frac{1}{12}\pi^2 x^2 + \cdots$

and Γ_0 is the collision rate without real or virtual infrared gravitons.

Using 4.1, we can get the power spectrum. Then power is

$$P(\leq \Lambda) = \int_0^\Lambda E d\Gamma(\leq E) \tag{4.2}$$

Differentiating Eq. 4.1, then Eq. 4.2 becomes

$$P(\leq \Lambda) = (B/B+1) b(B) \Lambda \Gamma_0 \tag{4.3}$$

B is always tiny ($\leq 10^{-38}$), then $B/(B+1) \cong 1$ and $b(B) \cong 1$. Then Eq. 4.3 becomes

$$P(\leq \Lambda) = B\Lambda\Gamma_0 \tag{4.4}$$

In this section, using Eq. 4.4 we calculate the power for non-relativistic collision. So we should get B, and determine Λ that has meaning for 'soft.'

First of all, we want to write B as

$$B = \frac{G}{\pi} \sum_{n,m} \eta_n m_n \eta_m m_m f(\beta_{nm}) \tag{4.5}$$

where

$$f(\beta) \equiv \frac{1+\beta^2}{2\beta(1-\beta^2)^{1/2}} ln[\frac{1+\beta}{1-\beta}]^{1/2} \tag{4.6}$$

and if n is final(initial) particle, then $\eta_n = +1(-1)$. β_{nm} is the relative velocity of particle n and m.

$$\beta_{nm} \equiv [1 - m_n^2 m_m^2/(p_n \cdot p_m)^2]^{1/2} \tag{4.7}$$

We treat the non-relativistic particles, then β_{nm} is expended in power serises of $\vec{v_n}$ and $\vec{v_m}$, with $\vec{v} \equiv \vec{p}/E$. Then

$$\beta_{nm}^2 = \vec{v_n}^2 + \vec{v_m}^2 + 2\vec{v_n} \cdot \vec{v_m} - \vec{v_n}^2 \vec{v_m}^2 - 3(\vec{v_n} \cdot \vec{v_m})^2 + 2(\vec{v_n}^2 + \vec{v_m}^2)(\vec{v_n} \cdot \vec{v_m}) + \cdots . \quad (4.8)$$

$f(\beta)$ can be expanded in powers of β^2,

$$f(\beta) = 1 + (11/6)\beta^2 + (63/40)\beta^4 + \cdots \quad (4.9)$$

Inserting Eq. 4.8 into 4.9,

$$f(\beta_{nm}) = 1 + (11/6)(\vec{v_n}^2 + \vec{v_m}^2 - 2\vec{v_n} \cdot \vec{v_m}) + (63/40)(\vec{v_n}^2 + \vec{v_m}^2)^2 - (79/30)(\vec{v_n}^2 + \vec{v_m}^2) \times (\vec{v_n} \cdot \vec{v_m}) + (4/5)(\vec{v_n} \cdot \vec{v_m})^2 + \cdots \quad (4.10)$$

Considering collision process, the energy and momentum are conserved

$$\sum_n \eta_n m_n (1 + \frac{1}{2}\vec{v_n}^2 + \frac{3}{8}\vec{v_n}^4 + \cdots) = 0 \quad (4.11)$$

$$\sum_n \eta_n m_n \vec{v_n} (1 + \frac{1}{2}\vec{v_n}^2 + \cdots) = 0 \quad (4.12)$$

Using Eq. 4.10, Eq. 4.11, and Eq. 4.12, Eq. 4.5 becomes up to order v^4,

$$B = (G/\pi)[(16/5)Q_{ij}Q_{ij} + (94/15)(Q_{ii})^2] \quad (4.13)$$

where

$$Q_{ij} = \frac{1}{2} \sum_n \eta_n m_n \vec{v_{ni}} \vec{v_{nj}} \tag{4.14}$$

The repeated indices are summed over 1,2,3. We treat non-relativistic collision, so the velocities in Eq. 4.14 should be subjected to the non-relativistic conservation laws

$$\sum_n \eta_n m_n (1 + \frac{1}{2} \vec{v_n}^2) = 0 \tag{4.15}$$

$$\sum_n \eta_n m_n \vec{v_n} = 0 \tag{4.16}$$

Using Eq. 4.15, Q_{ii} becomes

$$Q_{ii} = -\sum_n \eta_n m_n \tag{4.17}$$

And Eq. 4.16 makes Q_{ij} is invariant under the Galilean transformation, we can compute B in any convenient reference frame.

Let's think about the two-body scattering. Q_{ii} and Q_{ij} are

$Q_{ij} Q_{ij} = \frac{1}{2} \mu^2 v^4 sin^2\theta_c, \quad Q_{ii} = 0$

where μ is the reduced mass, $v = |v_1 - v_2|$ is the relative velocity, and θ_c is the scattering angle in the center-of-mass frame. Then B is

$$B = (8G/5\pi)\mu^2 v^4 sin^2\theta_c \tag{4.18}$$

We know that the rate for collisions per cm^3/sec is $vn_1n_2(d\sigma/d\Omega)$, where $n_i(i = 1, 2)$ is the number density of particle 1. Then the power for 1-2 collisions is

$$P(\leq \Lambda) = \frac{8G}{5\pi}\mu^2 v^5 n_1 n_2 V \Lambda \int \frac{d\sigma}{d\Omega} sin^2\theta_c d\Omega \tag{4.19}$$

with V the volume of the source. As mentioned before, Λ should be determined for having meaning 'soft.' For practical purposes, we define 'soft' radiation by taking Λ at half of the relative kinetic energy

$$\Lambda \approx \frac{1}{4}\mu v^2 \tag{4.20}$$

In the sun, the most frequent collisions are the Coulomb scattering between electrons and protons or electrons. In this case

$$\begin{aligned} \mu = m_e, \quad v = (3KT/m_e)^{1/2} \\ n_1 = n_e, \quad n_2 = n_e + n_p = 2n_e \end{aligned} \tag{4.21}$$

And the integral in Eq. 4.21 is the diffusion constant, this can be estimated as

$$\int \frac{d\sigma}{d\Omega} sin^2\theta_c d\Omega = \frac{8\pi e^2}{(3KT)^2} ln\Lambda_D \tag{4.22}$$

where Λ_D is the ratio of the Debye shielding radius to the average impact parameter. Then the power durring thermal colisions in the sun is

$$P_\odot = (32/5)G(3KT)^{3/2} m_e^{-1/2} n_e^2 V_\odot e^4 (\hbar c^5)^{-1} ln\Lambda_D \tag{4.23}$$

We did use the natural units, but in Eq. 4.23 we convert the power to cgs units. The parameters in the sun's core are

$$\begin{aligned} T &\cong 10^7 K \\ n_e &\cong 3 \times 10^{25} cm^{-3} \\ V_\odot &\cong 2 \times 10^{31} cm^3 \\ ln\Lambda_D &\cong 4. \end{aligned} \tag{4.24}$$

Then the solar gravitational radiation power is

$$P_\odot \cong 6 \times 10^{14} erg/sec. \tag{4.25}$$

This verify that the gravitational radiation during the thermal collisions is a stronger source than the classical source such as planetary motion. If a planet of mass m moving in a circular orbit of radius R around a star of mass M emits gravitational radiation with power

$$P = (32/5)Gc^{-5}m^2R^4(GM/R^3)^3. \tag{4.26}$$

For the Jupiter-Sun system this is $7.6 \times 10^{11} erg/sec.$ Venus and the Earth radiate comparable amounts, and the power for the other planets is considerably less. So the thermal gravitational radiation from the Sun is the dominant source of gravitational radiation from the solar system.

Bibliography

[1] S. Weinberg, "Photons and Gravitons in S -Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass," *Physical Review*, vol. 135, pp. 1–8, Aug. 1964. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRev.135.B1049

[2] ——, "Infrared Photons and Gravitons," *Physical Review*, vol. 140, pp. 1–9, Oct. 1965. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRev.140.B516

[3] ——, "Photons and Gravitons in Perturbation Theory: Derivation of Maxwell's and Einstein's Equations," *Physical Review*, vol. 138, pp. 1–15, May 1965. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRev.138.B988

초록

양자전기역학(Quantum electrodynamics)에서 매우 작은 운동량을 가진 광자(Soft Photon)를 다룰 때 발생하는 적외선 발산(Infrared divergence)을 radiative correction을 통해 제거하는 방식을 똑같이 사용하여 quantum theory of gravitation에서도 매우 작은 운동량을 가진 중력자(Soft Graviton)를 다룰 때 발생하는 적외선 발산(Infrared divergence)을 제거할 수 있음을 살펴볼 것이다. Non-Infrared External Particle의 질량이 0인 경우, Infrared Graviton을 다룰 때 발생하는 적외선 발산(Infrared divergence)이 에너지-운동량 보존 법칙을 통하여 제거됨을 확인할 것이다. Graviton Bremsstrahlung으로부터 유도된 공식으로 태양에서의 열적 충돌로 발생하는 중력파의 방출(Gravitational radiation)을 계산해볼 것이다.

주요어: 중력자, 적외선 발산, 중력파의 방출

학번: 2011-20390

Acknowledgements

이렇게 석사 과정을 마치고 졸업하게 되어 기쁘고 감사하게 생각합니다. 먼저는 저의 부모님께 어려운 사정에도 불구하고 공부를 해볼 수 있는 기회를 주셔서 감사하다는 말씀을 전하고 싶습니다. 또한 어려운 시간들을 보낸 동생 김지담에게도 고맙고 힘내라는 말을 전하고 싶습니다. 여러가지 힘든 시간들을 겪었지만 모두 앞으로 다가올 일들에 대해서도 겁먹지 않고 나아갈 수 있는 원동력이 될 것이라 생각합니다.
대학원 시절에 가장 감사드리고 싶은 분은 지도교수님이신 이수종 교수님 입니다. 많은 것을 이루셨음에도 끊임없는 호기심과 열정은 저로 하여금 학문에 대한 올바른 태도를 알 수 있게 해주셨습니다. 열정에서 머무는 것이 아니라 끊임없는 개인의 노력이 필요하며, 그 노력이 뒷받침 된 상태에서 수많은 커뮤니케이션이 있다면 많은 것을 얻을 수 있게 된다는 것을 알게된 것입니다. 비록 지금은 계속해서 공부를 할 수 없는 상황이지만, 사회에 나가서도 이는 적용될 것이라고 믿습니다. 또한 다시금 공부를 시작하게 되는 날이 온다면 꼭 기억하고 지켜나갈 것입니다.
대학원 생활을 함께한 선배, 동료, 후배들에게도 감사의 말을 전하고 싶습니다. 지금은 졸업했지만 많은 조언을 해주었던 재성오빠, 따뜻하게 대해줬던 승호오빠와 재원이, 많은 질문에 대해서 친절하게 알려주었던 진범오빠, 유쾌했던 상규오빠에게 감사의 말을 전합니다. 또한 동기인 우현이에게도 많은 도움을 받았던 것 같습니다. 동기로서 더 많이 챙겨주지 못해 미안한 점도 있지만 잘 해낼 것이라고

믿고 응원할 것입니다. 많은 시간 선배로서 도움을 주진 못했지만 후배인 수봉이, 명재, 준영이, 국현이에게도 감사하다는 말을 꼭 전하고 싶습니다. 또한 논문을 작성하는 데 도움을 준 강민에게도 감사하다는 말을 전합니다.
이렇게 많은 분들의 도움이 있어 제가 석사과정을 잘 마칠 수 있게 되었다고 생각합니다. 위의 분들과, 감사의 말을 따로 적진 못한 주위의 모든 분들에게도 감사의 말씀을 전합니다.